

CONSTRAINED WILLMORE TORI AND ELASTIC CURVES IN 2-DIMENSIONAL SPACE FORMS

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ABSTRACT. In this paper we consider two special classes of constrained Willmore tori in the 3-sphere. The first class is given by the rotation of closed elastic curves in the upper half plane - viewed as the hyperbolic plane - around the x -axis. The second is given as the preimage of closed constrained elastic curves, i.e., elastic curves with enclosed area constraint, in the round 2-sphere under the Hopf fibration. We show that all conformal types can be isometrically immersed into S^3 as constrained Willmore (Hopf) tori and write down all constrained elastic curves in H^2 and S^2 in terms of the Weierstrass elliptic functions. Further, we determine the closing condition for the curves and compute the Willmore energy and the conformal type of the resulting tori.

1. INTRODUCTION

Let $f : M \rightarrow S^3$ be a conformally immersed compact surface. It is called constrained Willmore if it is a critical point of the Willmore energy $\int_M (H^2 + 1) dA$ under conformal variations. The minimizer of the Willmore energy for a fixed conformal class can be viewed as the optimal realization of the underlying Riemann surface in three space. Such a minimizer exists for $M = \mathbb{C}/\Gamma$, see [17], if the underlying conformal class provides a immersion to S^3 with Willmore energy below 8π . Further, the minimizer is smooth and constrained Willmore. It is an open question whether the infimum of the Willmore energy is below 8π for every conformal class.

The global minimizer of the Willmore energy in the class of tori is the Clifford torus, see [22]. Further, [23] have shown that the homogenous tori T_r are minimizers of their respective conformal classes near the Clifford torus. For rectangular conformal classes the minimizers are conjectured to be the 2-lobed tori of revolution, which have constant mean curvature in S^3 , see figure 1. The Willmore energy of this family is monotonic increasing with the conformal type, see [18], and converges to 8π . The limiting surface is a double covering of a geodesic sphere. Thus the minimizer of the Willmore energy for tori with prescribed rectangular conformal class exists by [17]. Tori of revolution can be constructed by rotation of a closed curve in the upper half plane around the x -axis. The torus is constrained Willmore if and only if the curve is elastic in the upper half plane viewed as H^2 . Since [3] have shown that all embedded CMC tori are rotational, the pictured tori are the minimizers of the Willmore energy in their respective conformal classes restricted to CMC tori. For non rectangular conformal classes there are no candidates known in the literature, since tori of revolution are always of rectangular conformal types.

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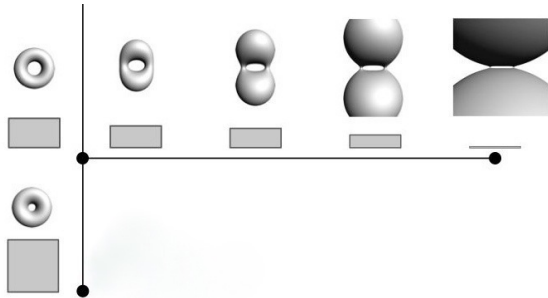


FIGURE 1. Embedded two lobed CMC tori of revolution in S^3 . (by Nick Schmitt)

First examples of Willmore tori, which are not minimal in a space form were found in by [25] in the class of Hopf tori. These are given the preimage of closed curves in S^2 under the Hopf fibration. The torus is (constrained) Willmore, if and only if the corresponding curve is (constrained) elastic, i.e., critical points of the energy functional with prescribed length and enclosed area. In contrast to tori of revolution [25] shows that all conformal types can be realized algebraically as Hopf tori. In the literature there exist an alternative notion of constrained Willmore surfaces. These are critical points of the Willmore functional with prescribed enclosed volume and surface area (Helfrich model). Since Hopf tori are flat and the mean curvature of the torus is simply the geodesic curvature of the curve in S^2 , constrained Willmore Hopf tori are constrained Willmore in both sense.

In this paper we study the two classes of constrained Willmore tori which comes from closed elastic curves in H^2 and closed (constrained) elastic curves in S^2 . We first show that every conformal class can be realized as a constrained Willmore (Hopf) torus via the direct method of calculus of variations. This generalizes the result by [25]. Then we derive explicit formulas for (constrained) elastic curves in 2-dimensional space forms. By viewing H^2 and S^2 as subsets of \mathbb{CP}^1 , we define the Schwarzian derivative q as a Möbius invariant of the curve. The curve is constrained elastic if and only if q is stationary under the first order KdV flow. Thus it is generically given in terms of a Weierstrass \wp -function. The \wp -function is well defined on a torus. This torus is our setup the spectral curve. We compute the closing conditions for the curves and show that every constrained elastic curve is isospectral to an elastic curve. Then we give formulas for the Willmore energy and the conformal type of the resulting torus.

In their paper [20] Langer and Singer constructed elastic curves in S^2 and H^2 without the enclosed area constraint. Our result is a generalization of this and uses the Schwarzian derivative instead of the geodesic curvature of the curve.

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2. EQUIVARIANT TORI IN THE 3-SPHERE

We consider $S^3 \subset \mathbb{C}^2$.

Definition. A map $f : \mathbb{C} \rightarrow S^3$ is called \mathbb{R} -equivariant, if there exist group homomorphisms

$$\begin{aligned} M : \mathbb{R} &\rightarrow \{\text{Möbius transformations of } S^3\}, t \mapsto M_t, \\ \tilde{M} : \mathbb{R} &\rightarrow \{\text{conformal transformations of } \mathbb{C}\}, t \mapsto \tilde{M}_t, \end{aligned}$$

such that

$$f \circ \tilde{M}_t = M_t \circ f, \text{ for all } t.$$

If f is doubly periodic with respect to a lattice $\Gamma \subset \mathbb{C}$, then f is a torus and the following proposition holds.

Proposition 1. *Let $f : T^2 \cong \mathbb{C}/\Gamma \rightarrow S^3$ be a equivariant conformal immersion. Then there exist a holomorphic coordinate $z = x + iy$ of T^2 together with $m, n \in \mathbb{N}$ and $\gcd(m, n) = 1$ such that*

$$f(x, y) = \begin{pmatrix} e^{imx} & 0 \\ 0 & e^{inx} \end{pmatrix} f(0, y).$$

The curve $\gamma(y) := f(0, y)$ (not necessarily closed) is called *profile curve of the surface*.

In this paper we only consider two very special cases of equivariant tori, namely the case of tori of revolution ($m = 1, n = 0$) and Hopf tori ($m = n = 1$).

Definition. Let M be a compact surface and let $f : M \rightarrow S^3$ be an immersion into the round sphere. The *Willmore energy* of f is defined to be

$$\mathcal{W}(f) = \int_M (H^2 + 1) dA,$$

where H is the mean curvature of f and dA is induced volume form.

A conformal immersion $f : M \rightarrow S^3$ is called *Willmore*, if it is a critical point of the Willmore energy W under all variations and it is called *constrained Willmore*, if it is a critical point of W under conformal variations, see [8] and [24].

It is shown in [19] that the Willmore functional reduces to the energy functional $\int_\gamma \kappa^2 ds$ for surfaces of revolution, where κ is the curvature of the profile curve in the hyperbolic plane, and s is the arc length parameter. Thus a surface of revolution is constrained Willmore if and only if its profile curve is elastic in H^2 . Further, [25] shows that the Willmore energy for a Hopf torus reduces to the generalized energy functional $\int_\gamma (\kappa^2 + 1) ds$ of the corresponding curve in S^2 . In particular the mean curvature of Hopf tori satisfies $H = \kappa$ and by construction the Gaussian curvature is zero. The conformal type of the torus is determined by the length and enclosed area of the curve. Thus a Hopf torus is constrained Willmore, if its profile curve is a critical point of the energy functional with fixed length and enclosed area.

Definition. Let γ be an arc length parametrized closed curve in a 2-dimensional space form and κ its geodesic curvature. The curve is called constrained elastic, if it is a critical point of the energy functional $\int_{\gamma} \kappa^2 ds$ with fixed length and enclosed area.

Proposition 2 ([8]). *Let γ be an arclength parametrized curve into a 2-dimensional space form of constant curvature G and let κ be its geodesic curvature in the space form. The Euler-Lagrange equation for a constrained elastic curve is:*

$$(2.1) \quad \kappa'' + \frac{1}{2}\kappa^3 + (\mu + G)\kappa + \lambda = 0,$$

for real parameters μ and λ .

This equation is the well known stationary first order modified Korteweg-de-Vries equation. The real parameters μ and λ are the length and respectively the enclosed area constraint for a closed curve. A solution to $\mu = \lambda = 0$ is a free elastic curve in the space form of curvature G . By multiplying the equation with $2\kappa'$ one can integrate the equation once and obtain

$$(2.2) \quad (\kappa')^2 = -\frac{1}{4}\kappa^4 - (\mu + G)\kappa^2 - 2\lambda\kappa - \nu.$$

Here ν is a real integration constant. We denote the polynomial on the right hand side by P_4 , i.e.,

$$P_4(x) := \frac{1}{4}x^4 + (\mu + G)x^2 + 2\lambda x + \nu$$

Theorem 1. *For given real numbers L and A satisfying the isoperimetric inequality on S^2*

$$L_0^2 - 4\pi A_0 - A_0^2 \geq 0,$$

there exist a smooth constrained elastic curve in S^2 minimizing the energy $\mathcal{E}(\gamma) = \int (\kappa^2 + 1)ds$ with length $L(\gamma) = L_0$ and enclosed area $A(\gamma) = A_0$.

Remark 1. We use the notion of oriented enclosed area of a curve in S^2 used in [25]. It is only well-defined modulo 4π .

Proof. The proof is a straight forward application of the direct method of calculus of variations. We want to find a minimizer of the Willmore energy in the set

$$\mathcal{S} := \{\gamma : S^1 \rightarrow S^2 \text{ smooth} \mid L(\gamma) = L_0 \text{ and } A(\gamma) = A_0\}.$$

By Theorem 1 of [25] the set is non empty, if the isoperimetric inequality holds. Thus $\mathcal{E}_0 := \inf\{E(\gamma) \mid \gamma \in \mathcal{S}\} \geq 0$. Without loss of generality we only consider arclength parametrized curves. Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{S} such that

$$\lim_{n \rightarrow \infty} \mathcal{E}(\gamma_n) = \mathcal{E}_0.$$

Since we have

$$(2.3) \quad \begin{aligned} \int |\gamma_n'|^2 ds &= L_0 \\ \int |\gamma_n''|^2 ds &= \int (\langle \gamma_n'', N_n \rangle^2 + \langle \gamma_n'', \gamma_n \rangle^2) ds = \int (\kappa_n^2 + 1) ds, \end{aligned}$$

there exist a subsequence of $(\gamma_n)_{n \in \mathbb{N}}$ which converges weakly in $W^{2,2}$ and strongly in $W^{1,2}$ to a curve γ_0 . Thus $\gamma_0 \in \mathcal{C}^1$ and $(\gamma_n)_{n \in \mathbb{N}}$ and $(\gamma'_n)_{n \in \mathbb{N}}$ converges point wise. Further, by the Gauß-Bonnet theorem the enclosed area can be computed as $A(\gamma_n) = 2m\pi - \int_{\gamma} \kappa_n ds$, where m is the winding number of the curve. Thus γ_0 is a minimizer of \mathcal{E} for curves lying in

$$\tilde{\mathcal{S}} := \{\gamma : S^1 \rightarrow S^2, \gamma \in W^{2,2} | L(\gamma) = L_0 \text{ and } A(\gamma) = A_0\}.$$

For the regularity of γ_0 we rewrite the Euler-Lagrange equation. The Hopf fibration induces a S^1 -fiberbundle with canonical connection on S^3 . A conformal parametrization of the Hopf torus f_0 corresponding to γ_0 is obtained by taking the horizontal lift $\tilde{\gamma}_0$ of γ_0 as the profile curve of f_0 , see Proposition 1. Note that the horizontal lift is well defined for $W^{2,2}$ curves and preserves the regularity. Let (T, N, B) denotes the Frénet frame of $\tilde{\gamma}_0$. Then γ_0 is a constrained elastic curve in S^2 if and only if there exist real constants λ and μ such that the vector field

$$X = (\kappa^2 + \lambda)T + 2\kappa'N + (2\kappa + \mu)B$$

is parallel with respect to the Levi-Civita connection on S^3 . Thus κ is a BV function on a compact interval and therefore $\kappa \in L^\infty$. Thus one can use the Caldéron-Zygmund estimates and obtain smoothness for κ . \square

Corollary 1. *Every conformal class of the torus can be realized as a constrained Willmore immersion in the 3-sphere.*

Proof. By [25] the conformal type of a Hopf torus is given by $(L/2, A/2)$ and the region, where the isoperimetric inequality holds covers the whole moduli space of conformal structures of tori. \square

3. CONSTRAINED ELASTIC CURVES IN SPACE FORMS

Since the Willmore functional is Möbius invariant, it seems to be more natural to consider a Möbius invariant setup here. Thus we consider

$$\gamma : \mathbb{R} \rightarrow H^2, S^2, \mathbb{R}^2 \hookrightarrow \mathbb{CP}^1$$

via affine coordinates. The Möbius invariant of a map into \mathbb{CP}^1 is the Schwarzian derivative. It can be defined by the following construction as in [10]. Let γ be a curve in \mathbb{CP}^1 . To γ there exist a lift $\tilde{\gamma}$ to \mathbb{C}^2 (not necessarily closed) with respect to the canonical projection from \mathbb{C}^2 to \mathbb{CP}^1 . Further, there exist a complex function a with $\hat{\gamma} := a\tilde{\gamma}$ such that $\det_{\mathbb{C}}(\hat{\gamma}, \hat{\gamma}') = 1$. Thus $\hat{\gamma}''$ and $\hat{\gamma}$ are linear dependent over \mathbb{C} and there exist a the complex valued function satisfying

$$(3.1) \quad \hat{\gamma}'' + q\hat{\gamma} = 0.$$

Definition. The function q is called the Schwarzian derivative of γ .

The curve is uniquely determined up to Möbius transformations by q . A straight forward computation gives

Lemma 1. *Let γ be a regular and arclength parametrized curve in a 2-dimensional space form of constant curvature G and let κ be its geodesic curvature. Then the Schwarzian derivative q of γ is given by*

$$q = \frac{i\kappa'}{2} + \frac{\kappa^2}{4} + \frac{G}{4}.$$

Further, if γ is constrained elastic in the space form, i.e., κ is a real solution of the stationary mKdV equation (2.2) with real parameters λ, μ and ν , then q satisfies the stationary KdV equation

$$(q')^2 + 2q^3 + cq^2 + 2dq + e = 0,$$

with real parameters c, d and e given by

$$(3.2) \quad \begin{aligned} c &= \mu - \frac{G}{2} \\ d &= -\frac{\nu}{4} - \frac{G^2}{16} - \mu \frac{G}{4} \\ e &= cd + \frac{\lambda^2}{4} + \frac{\mu^2 G}{4} - \frac{\nu G}{4}. \end{aligned}$$

The transformation $\kappa \mapsto q$ of an arclength parametrized curve is a geometric version of the well-known Miura transformation. Let

$$\begin{aligned} g_2 &:= \frac{c^2}{12} - d = \frac{(\mu + G)^2}{12} + \frac{\nu}{4} \\ g_3 &:= -\frac{cd}{12} + \frac{e}{4} + \frac{1}{6^3}c^3 = \frac{1}{216}(\mu + G)^3 + \frac{1}{16}\lambda^2 - \frac{1}{24}\nu(\mu + G) \\ P_3 &:= 4x^3 - g_2x - g_3. \end{aligned}$$

If $D = g_3^2 - 27g_2^3 \neq 0$ then the differential equation

$$(3.3) \quad f'^2 = P_3(f)$$

defines a double periodic meromorphic function - the Weierstrass \wp function. Its periods ω_i are linearly independent over the reals, i.e., the ω_i generates a lattice Γ in \mathbb{C} , and \wp is a well-defined function on $T^2 = \mathbb{C}/\Gamma$. The equation (3.2) is then solved by

$$q(x) = -2\wp(x + x_0) - \frac{1}{6}c,$$

for some constant $x_0 \in \mathbb{C}_*$. We refer to [1] for details on the Weierstrass elliptic functions.

A necessary condition for q to be the Miura transformation of a real valued curvature function κ is that the lattice invariants g_2 and g_3 are real. We also need that $D \neq 0$ to obtain a well-defined \wp -function, i.e., the polynomial P_3 has only simple roots and the generators of the lattice Γ are linear independent over the reals. We deal with the case of P_3 having multiple roots in section 3.2. For real g_2 and g_3 , the lattice Γ is rectangular or its double covering is rectangular, depending on the sign of its discriminant.

Definition. A solution of equation (3.2) is called orbitlike, if $D := g_3^2 - 27g_2^3 < 0$ and wavelike, if $D > 0$. The polynomial P_3 has multiple roots if and only if $D = 0$.

A curve in \mathbb{CP}^1 with Schwarzian derivative q solving equation (3.2) can be written in terms of Weierstrass ζ and σ functions. The Weierstrass ζ -function is determined by $\zeta' = -\wp$ and $\lim_{z \rightarrow 0} \zeta(z)z = 1$ and the Weierstrass σ function is given by $\frac{\sigma'}{\sigma} = \zeta$ and $\sigma(0) = 0$. Again, we refer to [1] for the properties of these functions.

Theorem 2. *Let \tilde{q} be a solution of equation (3.2) with real parameters c, d, e . We define a family of curves $\hat{\gamma}_E = (\hat{\gamma}_E^1, \hat{\gamma}_E^2) \subset \mathbb{C}^2$, $E \in \mathbb{R}$ by*

$$(3.4) \quad \begin{aligned} \hat{\gamma}_E^1 &= \frac{\sigma(x + x_0 - \rho)}{\sigma(x + x_0)} e^{\zeta(\rho)(x+x_0)} \\ \hat{\gamma}_E^2 &= \frac{\sigma(x + x_0 + \rho)}{\sigma(x + x_0)} e^{\zeta(-\rho)(x+x_0)}, \quad \text{with } \wp(\rho) = E. \end{aligned}$$

Then $\hat{\gamma}_E$ induces a family of curves γ_E in \mathbb{CP}^1 with Schwarzian derivative $q = (\tilde{q} + \frac{1}{6}c - E)$, if E is not a branch point of \wp .

Proof. We have $\tilde{q} = -2\wp(s + s_0) - \frac{1}{6}c$. By construction the functions $\hat{\gamma}_E^i$, $i = 1, 2$ have no poles and no common zeros if they are linearly independent over \mathbb{C} . This is the case if and only if E is not a branch point of \wp . Then the curve is well defined. Since

$$(\gamma_E^i)'' - 2\wp(x + x_0)\gamma_E^i = E\gamma_E^i,$$

the stated q_E is the Schwarzian derivative of the curve γ_E . \square

Lemma 2. *Let g_2 and g_3 be real constants with $g_3^2 - 27g_2^3 \neq 0$. And let \wp be the Weierstrass function with respect to the lattice $\Gamma \subset \mathbb{C}$ given by the lattice invariants g_2 and g_3 . If $x_0 \in \mathbb{C} \setminus \{\frac{1}{2}\Gamma\mathbb{R}\}$, then there exist a function $\kappa : \mathbb{R} \rightarrow \mathbb{R}$ with*

$$\wp(x + x_0) = -i \frac{\kappa'(x)}{4} - \frac{\kappa(x)^2}{8} - b,$$

where b is a real constant. Moreover, κ is a stationary mKdV solution with coefficients determined by g_2, g_3 .

Proof. We will first proof that $\wp(x + x_0)$ has the right form. By differentiating equation (3.3) we get another differential equation for \wp , namely

$$(3.5) \quad \wp''(x + x_0) = 6\wp(x + x_0)^2 - \frac{1}{2}g_2.$$

Consider now only the points $z \in \mathbb{C}/\Gamma$ with $\wp - \bar{\wp} \neq 0$. A reformulation of our statement is:

$$\wp + \bar{\wp} = (\zeta - \bar{\zeta} + \text{const}_1)^2 + \text{const}_2,$$

with const_1 purely imaginary, const_2 real and ζ is the Weierstrass ζ -function. Then we can define

$$(3.6) \quad \kappa := -2i(\zeta - \bar{\zeta} + \text{const}_1).$$

With (3.3) and (3.5) we obtain

$$2(\bar{\wp} - \wp)^3 = (\wp'' + \bar{\wp}'')(\bar{\wp} - \wp) + (\wp')^2 - (\bar{\wp}')^2.$$

This is equivalent to

$$2(\bar{\wp} - \wp) = \frac{\wp'' + \bar{\wp}''}{\bar{\wp} - \wp} + \frac{(\wp')^2 - (\bar{\wp}')^2}{(\bar{\wp} - \wp)^2}.$$

By integration we get

$$2(\zeta - \bar{\zeta} + \text{const}_1) = \frac{\wp' + \bar{\wp}'}{\bar{\wp} - \wp},$$

with a purely imaginary integration constant const_1 . Thus

$$\wp' + \bar{\wp}' = 2(\bar{\wp} - \wp)(\zeta - \bar{\zeta} + \text{const}_1).$$

Integrate again we obtain

$$\wp + \bar{\wp} = ((\zeta - \bar{\zeta}) + \text{const}_1)^2 + \text{const}_2,$$

with a real integration constant const_2 . The functions \wp and $\bar{\wp}$ are holomorphic and anti-holomorphic, respectively, therefore we get that the derivative of \wp with respect to $z = x + iy$ and the derivative of $\bar{\wp}$ with respect to \bar{z} is the same as the derivative of \wp and $\bar{\wp}$ with respect to x . Thus by replacing \wp by $\wp(x + x_0)$ this proves the statement. Since all the functions we consider are continuous the equation above is still valid at the boundaries in the x -direction. Thus it is necessary to choose a x_0 which does not lie on the real axis or on a parallel translate of the real axis by a half lattice point. These choices of x_0 does not lead to an arclength parametrized constrained elastic curve, since q would be real valued.

Now we show that κ defined by equation (3.6) is mKdV stationary. We have $\wp = -i\frac{\kappa'(x)}{4} - \frac{\kappa(x)^2}{8} - b$ and therefore

$$\begin{aligned} \wp(x + x_0)'' &= -i\frac{1}{4}\kappa'''(x) - \frac{1}{4}\kappa''(x)\kappa(x) - \frac{1}{4}(\kappa'(x))^2 \\ 6\wp(x + x_0)^2 &= \frac{3i}{8}\kappa'\kappa^2 + 3ib\kappa' - \frac{3}{8}\kappa'^2 + \frac{3}{32}\kappa^4 + 6b^2 + \frac{3}{2}b\kappa^2. \end{aligned}$$

Hence the imaginary part of equation (3.5) yields

$$(3.7) \quad \kappa''' + \frac{3}{2}\kappa'\kappa^2 + 12b\kappa' = 0.$$

Thus κ is the curvature of a arclength parametrized constrained elastic curve. \square

Remark 2. Lemma 2 shows that the curve γ_E with Schwarzian derivative q defined in Theorem 2 is Möbius equivalent to an arc length parametrized constrained elastic curve γ in a 2-dimensional space form. We fix the Möbius transformation in section 3.4.

3.1. The roots of the polynomials P_3 and P_4 . Since we want to consider closed curves, the curvature function κ is periodic and achieves its maximum and minimum. Thus we can always choose $\kappa'(0) = 0$ as initial value for the equation (2.1). This corresponds to the choice of $x_0 \in i\mathbb{R} \setminus \{\frac{1}{2}\Gamma\}$. The necessary and sufficient condition for the existence of a real function κ solving equation (2.2) with parameters μ , λ and ν is that the polynomial P_4 has real roots. In the case of 4-th order polynomial there exist an algorithm to compute its roots explicitly. To P_4 one associate a polynomial

of degree 3 - the cubic resolvent. In our case it is the polynomial $P_3(s)$, with $s = 16x + \frac{8}{3}(\mu + G)$. The roots of $P_4(x)$ can then be computed out of the roots of $P_3(s)$. In particular, P_4 has multiple roots if and only if P_3 has multiple roots. The condition for $P_4(x)$ to have real roots is then:

Lemma 3. *Let $P_4(x)$ be the real polynomial of degree 4 given in (2.2) with only simple roots and let $P_3(s)$ denote its cubic resolvent. Then $P_4(x)$ has real roots if and only if all real roots of P_3 are non-negative.*

Proof. Let e_1, e_2 and e_3 be the roots of P_3 . Then the cubic resolvent can be written as $P_3(s) = (s - e_1)(s - e_2)(s - e_3)$. We obtain in our particular case that

$$P_3(s = 0) = P_3(x = \frac{1}{6}(\mu + G)) = -e_1 e_2 e_3 = -\frac{1}{16}\lambda^2 \leq 0.$$

For $D > 0$ there is only 1 real root and a pair of complex conjugate roots of P_3 . Therefore the real root must be non negative. For $D < 0$ all roots of $P_3(s)$ must be non-negative in order to obtain real roots of P_4 by the algorithm to compute the roots. \square

Remark 3. The proof shows that for given g_2 and g_3 and $(\mu + G)$ the parameter λ is fixed up to sign. The choice of the sign corresponds to the transformation $\kappa \mapsto -\kappa$.

Corollary 2. *The stationary mKdV equation (2.2) with real parameters $(\mu + G)$, λ and ν has real solutions if and only if $\frac{1}{6}(\mu + G)$ is less or equal to all real roots of the polynomial P_3 . Equality holds if and only if $\lambda = 0$.*

Corollary 3. *There exist no orbitlike free elastic curves on S^2 . Further, there are no orbitlike elastic curves corresponding to Willmore Hopf tori.*

Proof. In the case of $D < 0$ the condition for the existence of real solutions is computed to be

$$(\mu + G) < 0 \text{ and } \frac{1}{3}(\mu + G)^2 \leq \nu \leq (\mu + G)^2.$$

But for free elastic curves in S^2 we have: $G > 0$, and $\lambda = \mu = 0$ and for Willmore Hopf tori we have : $G > 0$, $\lambda = 0$ and $(\mu + G) = \frac{1}{2}G > 0$. \square

3.2. Multiple roots. We have shown that in the case where the polynomial P_4 has only simple roots the equation (3.2) can be solved using the Weierstrass \wp -function. Now we study the case where P_4 has multiple roots.

Since we are looking for periodic solutions, we can restrict ourselves without loss of generality to the initial value problem for equation (2.1) with initial values

$$\kappa(0) = \kappa_0 \quad \text{and} \quad \kappa'(0) = 0.$$

Then κ_0 is a real root of P_4 with parameters λ, μ and ν . There are two cases to consider. In the first case κ_0 is a multiple zero of P_4 itself. Then it is also a root of $\frac{\partial P_4}{\partial \kappa}$, which is the right hand side of equation (2.1). Therefore $\kappa \equiv \kappa_0$ is the unique solution to the given initial value problem by Picard-Lindelöf.

In the second case P_4 has multiple roots but κ_0 is a simple root of P_4 .

Definition. A solution of equation (2.1) (or of equation (3.2)), where P_4 has multiple roots and the initial condition κ_0 is a simple root is called an asymptotic solution.

Proposition 3. *Asymptotic solutions with $\lambda = 0$ are never periodic.*

Proof. For $\lambda = 0$ we have the differential equation

$$(\kappa')^2 = -\frac{1}{4}\kappa^4 - 2(\mu + G)\kappa^2 - \nu.$$

The polynomial on the right hand side is even and has multiple roots by assumption. In order to obtain non constant solutions we need at least 1 simple root of P_4 . By symmetry the only case to consider is that the multiple root of P_4 is at $\kappa = 0$ with multiplicity 2 and we have 2 simple roots for $\kappa = \pm\kappa_0$, and $\kappa_0 \in \mathbb{R}_+$.

We solve an initial value problem for the differential equation of second order

$$\kappa'' + \frac{1}{2}\kappa^3 + (\mu + G)\kappa = 0,$$

with initial value $\kappa(0) = \kappa_0$ and $\kappa'(0) = 0$. At $\kappa(0)$ we obtain that $\kappa''(0) = \frac{\partial(\kappa')^2}{\partial\kappa}|_{x=0} < 0$. Thus there exist an $\epsilon > 0$ with $\kappa'(t) < 0$ for $t \in (0, \epsilon)$ and the curvature function κ decreases monotonically for $t \in (0, \epsilon)$. Let $T := \sup\{\epsilon \in \mathbb{R}_+ | \kappa'(t) < 0 \text{ for } t \in (0, \epsilon)\}$. If $T < \infty$, then $\kappa'(T) = 0$ and we obtain $\kappa(T)$ is a root of P_4 . Since κ is continuous, we obtain $\kappa(T) = 0$, which is a multiple root. By Picard-Lindelöf we get that $\kappa(t) \equiv 0$ is the unique solution to the initial value problem $\kappa'(T) = \kappa(T) = 0$. This contradicts $\kappa(0) = \kappa_0 \neq 0$. Therefore $T = \infty$ and κ is not periodic. \square

Corollary 4. *Constrained Willmore tori of revolution and Willmore Hopf tori are either homogenous, i.e., $\kappa \equiv \kappa_0$ is constant, or P_4 has only simple roots.*

Remark 4. Closed asymptotic solutions do exist for curves in S^2 . These are obtained by a simple factor dressing of a multi-covered circle. In fact one can show that all asymptotic solutions on S^2 can be obtained this way.

3.3. Closing Conditions. To obtain closing conditions for the curves γ_E defined in Theorem 2 we compute their monodromy. The curve γ_E closes if and only if the monodromy is a rotation by a rational angle. We fix a lattice Γ in \mathbb{C} with real lattice invariants g_2 and g_3 and get a \wp -function with respect to this lattice. We denote by ω_i , $i = 1, 2, 3$, the half periods of Γ and fix ω_1 to be the half period lying on the real axis. For real g_2 and g_3 we always obtain a half lattice point on the imaginary axis, which we denote by ω_3 . In the case of $D > 0$ we have $\omega_1 = \omega_3 \bmod \Gamma$.

Proposition 4. *With the notations above the curve γ_E closes after n periods of the Weierstrass \wp -function if and only if there exist a $m \in \mathbb{N}$ with $\gcd(m, n) = 1$ such that*

$$2\eta_1\rho - 2\zeta(\rho)\omega_1 = \frac{m}{n}\pi i.$$

Here ζ is the Weierstrass ζ -function, $\eta_1 := \zeta(\omega_1)$ and $E = \wp(\rho)$.

Remark 5. Geometrically speaking, the number m is the winding number of the curve and the number n the lobe number.

Proof. Provided that E is not a branch point of the \wp -function the curve $\gamma_E = [\hat{\gamma}_E^1, \hat{\gamma}_E^2]$ is given by two complex valued functions

$$\begin{aligned}\hat{\gamma}_E^1 &= \frac{\sigma(x + x_0 - \rho)}{\sigma(x + x_0)} e^{\zeta(\rho)(x+x_0)} \\ \hat{\gamma}_E^2 &= \frac{\sigma(x + x_0 + \rho)}{\sigma(x + x_0)} e^{\zeta(-\rho)(x+x_0)}, \quad \text{with } \wp(\rho) = E.\end{aligned}$$

Further let ζ be the Weierstrass ζ -function and define $\eta_1 := \zeta(\omega_1)$. This is real because the lattice invariants g_2 and g_3 are real. With the formulas for the monodromy of the Weierstrass σ function we obtain:

$$\begin{aligned}\hat{\gamma}_E^1(x + 2\omega_1) &= e^{-2\eta_1\rho + 2\zeta(\rho)\omega_1} \hat{\gamma}_E^1(x) \\ \hat{\gamma}_E^2(x + 2\omega_1) &= e^{2\eta_1\rho - 2\zeta(\rho)\omega_1} \hat{\gamma}_E^2(x).\end{aligned}$$

The monodromy of the γ_E is the quotient of the both monodromies computed here. Therefore we get that the curve closes after n periods if and only if there exist a $m \in \mathbb{Z}$ with (m, n) coprime such that

$$e^{4\eta_1\rho - 4\zeta(\rho)\omega_1} = e^{\frac{m}{n}2\pi i},$$

which proves the statement. \square

Corollary 5. *Varying x_0 yields isospectral deformations of constrained elastic curves, i.e., deformations preserving the monodromy and the parameters g_2 , g_3 and E . In particular, every constrained elastic curve is isospectral to an elastic curve, i.e., a solution of equation (2.1) with $\lambda = 0$.*

Proof. For $\lambda = 0$ we have that $\frac{1}{6}(\mu + G)$ is the smallest real root of $P_3(x)$. Thus we obtain $\frac{1}{6}(\mu + G) = \wp(\omega_3)$ and $\nu = 4g_2 - 3\wp(\omega_3)^2$. The roots of P_4 and therefore the possible choices of κ_0 are

$$\begin{aligned}\kappa_0^1 &= \sqrt{12\wp(\omega_3) + \sqrt{156\wp(\omega_3) - 16g_2}} \\ \kappa_0^2 &= -\sqrt{12\wp(\omega_3) + \sqrt{156\wp(\omega_3) - 16g_2}} \\ \kappa_0^3 &= \sqrt{12\wp(\omega_3) - \sqrt{156\wp(\omega_3) - 16g_2}} \\ \kappa_0^4 &= -\sqrt{12\wp(\omega_3) - \sqrt{156\wp(\omega_3) - 16g_2}},\end{aligned}\tag{3.8}$$

if the solution is orbitlike. For wavelike solutions the possible choices of κ_0 are only κ_0^1 and κ_0^2 . And the possible values of $\wp(x_0)$ are

$$\wp(x_0^1) = -2\wp(\omega_3) - \frac{1}{8}\sqrt{156\wp(\omega_3) - 16g_2}, \quad \wp(x_0^2) = -2\wp(\omega_3) + \frac{1}{8}\sqrt{156\wp(\omega_3) - 16g_2}.$$

By construction only the first choice corresponds to $x_0 \in i\mathbb{R}$. Therefore there exist a unique $x_0 \in i(0, -i\omega_3)$ such that κ_0 is a root of P_4 . \square

Because of the above corollary, we restrict ourselves in the following to the case with $\lambda = 0$.

Theorem 3. *Let g_2 and g_3 be real constants with $g_3^2 - 27g_2^3 \neq 0$. Then every rational point of the function*

$$g : i\mathbb{R} \setminus \{\omega_3\mathbb{Z}\} \rightarrow i\mathbb{R}, g(\rho) = \eta_1\rho - \zeta(\rho)\omega_1$$

gives rise to a closed elastic curve γ_E , $E = \wp(\rho)$, as defined in Theorem 2 on a round S^2 with curvature $G = 4(\wp(\omega_3) - E)$. In particular, for fixed g_2 and g_3 there exist to every integer n a simply closed elastic curve with n lobes.

Proof. The polynomial P_3 defining the Weierstrass \wp -function has either 1 or 3 real roots. By assumption $\wp(\omega_3) = \frac{1}{6}(\mu + G)$, where $\omega_3 \in i\mathbb{R}$ is a half lattice point of Γ . We vary ρ , with $\wp(\rho) = E$, to close the curves. Since $E = \frac{1}{6}(\mu - \frac{1}{2}G)$, we obtain $\rho \in i\mathbb{R} \setminus \{\omega_3\mathbb{Z}\}$, see [1]. For fixed real invariants g_2 and g_3 we get that η_1 and ω_1 are also real. Further, for $\rho \in i\mathbb{R}$ the constant $\zeta(\rho) \in i\mathbb{R}$, too. Thus the map

$$g : i\mathbb{R} \rightarrow i\mathbb{R}, g(\rho) = \eta_1\rho - \zeta(\rho)\omega_1,$$

is well defined and $g(i\mathbb{R})$ is a nontrivial interval since

$$\lim_{\rho \rightarrow \pm\infty} g(\rho) = \pm\infty \text{ and } g(\omega_3) = 0 \text{ or } g(\omega_3) = \frac{1}{2}\pi i,$$

depending on whether the solution is wavelike or orbitlike. \square

Remark 6. For constrained elastic curves in S^2 it is necessary to choose $\rho \in i\mathbb{R} \bmod \Gamma$. Thus it is isospectral to an elastic curve in a space form of positive curvature and $\frac{1}{6}(\mu + G) > E = \frac{1}{6}(\mu - \frac{1}{2}G)$. Nevertheless, by decreasing $\frac{1}{6}(\mu + G)$ for fixed g_2 , g_3 and E^1 , the resulting curves first become a constrained elastic curve in \mathbb{R}^2 for $\frac{1}{6}(\mu + G) = E$ and then turns into a constrained elastic (but not elastic) curve in H^2 .

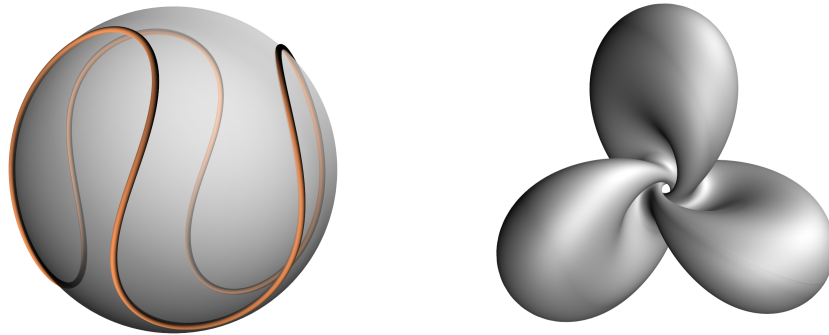


FIGURE 2. Wavelike elastic curve in S^2 to parameters $\mu = -\frac{1}{2}$ and $\lambda = 0$ in S^2 and corresponding Willmore Hopf torus.(by Nick Schmitt)

¹By choosing $\frac{1}{6}(\mu + G)$ according to Lemma 2 the parameter λ is determined up to sign and ν is fixed and there is a $x_0 \in i(0, -i\omega_3)$ with $\wp(x_0) = -\frac{\kappa_0^2}{8} - \frac{1}{12}(\mu + G)$. Therefore varying $\frac{1}{6}(\mu + G)$ is equivalent to the isospectral deformations given by varying x_0

Proposition 5. *Let g_2 and g_3 be real constants with $g_3^2 - 27g_2^3 > 0$ and γ_E be the family of curves defined in Theorem 2. Then there exist at most one closed elastic curve a space form of constant curvature $G < 0$ in that family.*

Proof. In this case $e = \frac{1}{6}(\mu + G)$ is the only real root of P_3 . Further ρ with $\wp(\rho) = E > e$ does not lie on the imaginary axis. Since E must be real, we get $\rho \in \mathbb{R}$ and thus $\zeta(\rho) \in \mathbb{R}$. Therefore the only chance to get a closed solution is that

$$\rho\eta_1 - \zeta(\rho)\omega_1 = 0.$$

The solution holds obviously for $\rho = \omega_1$ but this choice contradicts the fact that $E > e$. The closing condition can be interpreted as the intersection of the line given by $\rho \mapsto \rho \frac{\eta_1}{\omega_1}$ with the graph of the function $\zeta|_{\mathbb{R}}$. The function $\zeta|_{\mathbb{R}}$ is anti-symmetric with respect to ω_1 and has a simple pole in 0 and is convex for $\rho < \omega_1$ and concav for $\rho > \omega_1$. Thus there exist two other intersection points if and only if $-\wp(\omega_1) = -(E + \frac{1}{4}G) > \frac{\eta_1}{\omega_1}$, which makes the same curve. Otherwise there are no other intersection points and no closed curves. \square

Example 1. A closed curve in this class is a elastic figure-eight in H^2 . It is shown in [20] that there is no free elastic wavelike curve in the hyperbolic plane. Thus there is no Willmore torus coming from this construction.

Theorem 4. *Let g_2 and g_3 be real constants with $g_3^2 - 27g_2^3 < 0$. Then every rational point of the function*

$$g : i\mathbb{R} \setminus \{\omega_3\mathbb{Z}\} \rightarrow i\mathbb{R}, g(\tilde{\rho}) = \eta_1(\tilde{\rho} + \omega_1) - \zeta(\tilde{\rho} + \omega_1)\omega_1$$

gives rise to a closed constrained elastic curve γ_E ($E = \wp(\rho)$) as defined in Theorem 2) in H^2 with curvature G . In particular, for fixed g_2 and g_3 there exist to every integer $n > 1$ a simply closed elastic curve with n lobes.

Proof. The polynomial P_3 has three real roots and thus we can choose a $E > \frac{1}{6}(\mu + G)$ such that $P_3(E) < 0$ by varying $G < 0$. The corresponding ρ satisfies $\rho = \tilde{\rho} + \omega_1$ with $\tilde{\rho} \in i\mathbb{R}$ and

$$\overline{\zeta(\tilde{\rho} + \omega_1)} = -\zeta(\tilde{\rho} - \omega_1) = -\zeta(\tilde{\rho} + \omega_1) + 2\eta_1.$$

Thus the function

$$g(\tilde{\rho}) = \eta_1(\tilde{\rho} + \omega_1) - \zeta(\tilde{\rho} + \omega_1)\omega_1$$

is purely imaginary. Further $g(\omega_3) = \frac{1}{2}\pi i$ and $g(0) = 0$. By the same argument as in Theorem 3 we get a dense set of solutions. In particular, for $n > 1$ we obtain $\frac{1}{2n}\pi i \in g(i\mathbb{R})$. \square

Remark 7. In contrast to elastic curves in S^2 , elastic curves in \mathbb{H}^2 are never isospectral to constrained elastic curves in other space forms.

3.4. How to obtain the Space form. We want to show that the curves stated in Theorem 2 are already the constrained elastic curves we are looking for without applying any Möbius transformations. We use the Poincare disc model or the upper half plane model of $H^2 \hookrightarrow \mathbb{C}$ (depending whether the function g defined above is real or imaginary valued) and consider $S^2 = \mathbb{C} \cup \{\infty\}$. The curve γ_E given by Theorem 2 is Möbius equivalent to a constrained elastic curve γ in a space form \mathcal{G} of constant curvature G . A Möbius transformation M is fixed by its values on 3 points. We want to determine the Möbius transformation M from \mathcal{G} to \mathbb{CP}^1 which maps the arclength parametrized constrained elastic curve γ to γ_E . Without loss of generality we can fix $\gamma(0) \in i\mathbb{R}$.

For real valued parameter E the function g is either real or imaginary valued. In the first case the monodromy is a rotation which has two fixed points 0 and ∞ in \mathbb{CP}^1 and this rotation must be an isometry of \mathcal{G} . Thus we use the Poincare disc model of the hyperbolic plane here. Since the inversion at the unit circle preserves the constrained elastic property of a curve, the only Möbius transformations left are $z \mapsto rz$, for a real number r . We can fix r by asking the curve γ_E to be arc length parametrized with respect to the induced metric (which we need only to check in one point), i.e., $|\gamma'_E(0)|_{\mathcal{G}}^2 = 1$. In the second case (which only happens for constrained elastic curves in H^2), the hyperbolic space is given by the upper half plane and again the arclength property fixes the parameter r . The choice of r corresponds to the choice of the infinity boundary of the hyperbolic plane or respectively the image of the geodesic under the stereographic projection of S^2 . If the space form is \mathbb{R}^2 , then multiplication with r preserves the constrained elastic property.

3.5. Constrained Willmore cylinders of revolution. Constrained Willmore cylinders of revolution have constant mean curvature in a 3-dimensional space form by [7]. The different choices of the Sym-point E used here correspond to the space forms in which the cylinders have constant mean curvature.

Proposition 6. *Let γ_E be a elastic curve in H^2 , as defined in Theorem 2 and f the corresponding constrained Willmore cylinder of revolution. Then f is CMC in H^3 with mean curvature $|H| < 1$ if and only if γ_E is wavelike. If γ is orbitlike we have the following:*

For $P_3(E) < 0$ the cylinder f is CMC in S^3 .

If $P_3(E) > 0$ f is CMC in H^3 with mean curvature $H > 1$.

Proof. The spectral curve of a constrained Willmore cylinder of revolution is determined by the kernel of the operator (see [14]).

$$D_1 = \partial_y + \begin{pmatrix} -ia & i\frac{\kappa}{2} \\ i\frac{\kappa}{2} & ia \end{pmatrix},$$

where $a \in \mathbb{C}$ is the spectral parameter and κ the geodesic curvature. The spectral curve has a holomorphic and an anti-holomorphic involution, which covers the involutions $\sigma : a \mapsto -a$ and $\rho : a \mapsto \bar{a}$ on the a -plane, respectively.

The spectral curve we use here, i.e., the torus defined by the parameters g_2 and g_3 , is obtained by considering the spectrum of the operator

$$D_2 = \partial_y + \begin{pmatrix} 0 & q - \eta \\ -1 & 0 \end{pmatrix},$$

where $\eta = E - \frac{1}{6}c$ and q is the Schwarzian derivative. We define a transformation of the spectral parameters $\eta = -a^2 - \frac{1}{4}$, which is a double covering of the η -plane by the a -plane branched at $\eta = -\frac{1}{4}$ and $\eta = \infty$. Further we have $q = i\frac{\kappa'}{2} + \frac{\kappa^2}{4} - \frac{1}{4}$. Then these operators are gauge equivalent and the gauge transformation from D_2 to D_1 is given by

$$g = \begin{pmatrix} -i\frac{\kappa}{2} - ia & -i\frac{\kappa}{2} + ia \\ 1 & 1 \end{pmatrix},$$

for $a \in \mathbb{C}_*^2$. The surface spectral curve is a hyperelliptic curve over the a -plane. Its branch points are determined by the branch points of the curve spectral curve and the branch points of the double covering of the parameters. The curve spectral curve has 4 branch points which are the branch points of the corresponding Weierstrass \wp -function. If any of these branch points coincides with the branch points of the parameter covering, then the branch point becomes a regular point of the spectral curve of D_1 . Otherwise every branch point of the spectral curve of D_2 makes 2 branch points of D_1 . The point $\eta = \infty$ is a common branch point of the spectral curve of D_2 and $\eta = -a^2 - \frac{1}{4}$, thus $a = \infty$ is not a branch point of the spectral curve of D_1 . The point $\eta = -1/4$ is also a branch point since $\frac{1}{6}(\mu - 1)$ is a branch point of the Weierstrass \wp -function.

Whether the constrained Willmore torus of revolution is a CMC torus in a space form depends on whether the involution $\rho \circ \sigma$ has fixed points. This is the case if and only if there are branch points of the surface spectral curve over $a \in i\mathbb{R}$, i.e., $\eta \in \mathbb{R}$ and $\eta \geq -\frac{1}{4}$. This happens if and only if the curve spectral curve is branched over $\eta \in \mathbb{R}$, $\eta > -\frac{1}{4}$.

For wavelike elastic curves the curve spectral curve has only 1 real branch point over $\eta = -\frac{1}{4}$, which vanishes over the parameter a . Therefore there is no branch point of the surface spectral curve over $a \in i\mathbb{R}$. Thus this case corresponds to CMC cylinders in H^3 with mean curvature $|H| < 1$, see [4]

For orbitlike elastic curves the curve spectral curve has 3 real roots and by Corollary 2. All roots are greater or equal to $\frac{1}{6}(\mu - 1)$. Thus the corresponding values of η are real and satisfy $\eta > -\frac{1}{4}$. Thus all branch points of the surface spectral curve lie over $a \in i\mathbb{R}$ and the involution $\rho \circ \sigma$ has fixpoints.

By the Sym-Bobenko formula, see [6], the surface is CMC in S^3 , if the Sym-points are fixed under the involution $\rho \circ \sigma$ (which happens for $P_3(E) < 0$) and the surface is CMC in H^3 if the Sym-points are no fix points of the involution ($P_3(E) > 0$). \square

3.6. Conformal Type and Willmore Energy. The conformal types of tori of revolution and Hopf tori in terms of their profile curve were derived in [19] and [25].

²The spectral curve is an analytic variety and it is thus determined by its generic points.

Theorem 5. *Let $f : T^2 \rightarrow S^3$ be either a constrained Willmore torus of revolution or a constrained Willmore Hopf torus determined by the formulas of Theorem 2 for fixed parameters $g_2, g_3, E \in \mathbb{R}$. Then we have the following.*

- *If f is a constrained Willmore torus of revolution, then its conformal class is given by the lattice generated by $z_1 = 2\pi$ and $z_2 = i\sqrt{G}L$ and its Willmore energy is given by*

$$\mathcal{W}(f) = 8n\eta_1\pi - 4n\omega_1\wp(\omega_3)\pi.$$

- *If f is a constrained Willmore Hopf torus, then its conformal class is given by the lattice generated by $z_1 = 2\pi$ and $z_2 = \frac{1}{2}GA + \frac{1}{2}i\sqrt{G}L$ and its Willmore energy is given by*

$$\mathcal{W}(f) = 16n\eta_1\pi - 8n\omega_1E\pi.$$

Here $L = 2n\omega_1$ denotes the length of the curve in the respective space form and A is the oriented enclosed area of the curve in S^2 is given by

$$\frac{1}{2}GA \bmod 2\pi = (2m\pi - 4inn\eta_1x_0 - 2n\omega_1(\frac{1}{2}\kappa(0) - 2i\zeta(x_0))) \bmod 2\pi.$$

Proof. Recall that for constrained Willmore tori of revolution and constrained Willmore Hopf tori we have

$$\wp(x + x_0) = \frac{1}{4}i\kappa' - \frac{1}{8}\kappa^2 - \frac{1}{12}(\mu + G),$$

where κ is the geodesic curvature of the arclength parametrized profile curve in the space form of curvature G . Thus the integral of the real part of the Weierstrass \wp function, i.e., the real part of Weierstrass ζ -function determines the bending energy of the curve. We have

$$\int_{\gamma} (\kappa^2 + \frac{2}{3}(\mu + G))ds = 8n(\operatorname{Re}(\zeta(x - x_0 + 2\omega_1) - \zeta(x - x_0))) = 16n\eta_1,$$

if the curve closes after n periods of \wp . For constrained Willmore tori of revolution the Willmore energy is given by

$$\mathcal{W}(f) = \frac{1}{2}\pi \int_{\gamma} \kappa^2 ds.$$

Since constrained Willmore tori of revolution comes from elastic curves in H^2 , we have $\wp(\omega_3) = \frac{1}{6}(\mu + G)$ and thus

$$\mathcal{W}(f) = 8n\eta_1\pi - 4n\omega_1\wp(\omega_3)\pi.$$

For constrained Willmore Hopf tori we have

$$\mathcal{W}(f) = \pi \int_{\gamma} (\kappa^2 + G)ds.$$

Since $E = \frac{1}{6}(\mu - \frac{1}{2}G)$ the Willmore energy of a constrained Willmore Hopf torus is computed to be

$$\mathcal{W}(f) = 16n\eta_1\pi - 8n\omega_1E\pi.$$

The conformal type is given by two vectors generating the lattice $\Gamma \in \mathbb{C}$. In the case of tori of revolution these are given by

$$z_1 = 2\pi \quad \text{and} \quad z_2 = i\sqrt{GL}$$

where L is the length of the curve in the space form of curvature $G < 0$. Since the profile curve γ_E is arclength parametrized, we get that the length of the curve is $2n\omega_1$.

For constrained Willmore Hopf tori the lattice is generated by

$$z_1 = 2\pi \quad \text{and} \quad z_2 = \frac{1}{2}GA + \frac{1}{2}i\sqrt{GL},$$

where A is the oriented enclosed area of the curve, see [25], which is only well defined modulo $\frac{1}{G}4\pi$. By the Gauß-Bonnet theorem the enclosed area of a curve is given by

$$GA = 2\pi m - \int_{\gamma} \kappa ds \bmod 4\pi,$$

where m is the winding number of the curve. On the other hand we have:

$\text{Im}\zeta(x + x_0) = \frac{\kappa}{4} - \frac{\kappa_0}{4} - i\zeta(x_0)$. Thus:

$$\begin{aligned} \frac{1}{2} \int_{\gamma} \kappa ds - 2n\omega_1(\frac{1}{2}\kappa(0) - 2i\zeta(x_0)) &= 2\text{Im}(\ln(\sigma(x + x_0 + 2n\omega_1)) - \ln(\sigma(x + x_0))) \\ &= -i\ln\left(\frac{e^{2n\eta_1(x+x_0+\omega_1)}\sigma(x+x_0)\sigma(x-x_0)}{e^{2n\eta_1(x-x_0+\omega_1)}\sigma(x+x_0)\sigma(x-x_0)}\right) \\ &= -i\ln(e^{4n\eta_1 x_0}). \end{aligned}$$

The logarithm is only well defined modulus $2\pi i$. We obtain

$$\frac{1}{2} \int_{\gamma} \kappa ds - 2n\omega_1(\frac{1}{2}\kappa_0 - 2i\zeta(x_0)) = (2\pi - 4ni\eta_1 x_0) \bmod 2\pi.$$

Therefore $\frac{1}{2}GA$ is given by:

$$(\pi m - 4in\eta_1 x_0 + 2n\omega_1(\frac{1}{2}\kappa_0 - 2i\zeta(x_0))) \bmod 2\pi.$$

□

REFERENCES

- [1] Abramowitz and Stegun (Eds.). *Weierstrass Elliptic and Related Functions*. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, pp. 627-671, 1972.
- [2] Ambrosio, Fusco and Pallara *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
- [3] Andrews and Li. *Embedded constant mean curvature tori in the three-sphere*. arXiv:1204.5007v2, 2012.
- [4] Babich and Bobenko *Willmore tori with umbilic lines and minimal surfaces in hyperbolic space*. Duke Math. J. Volume 72, Number 1, pp 151 - 185, 1993.
- [5] Bernard, Y. *Analysis of COnstrained Willmore Surfaces*. arXiv:1211.4455, 2012.
- [6] Bobenko, A. *Constant mean curvature surfaces and integrable equations*. Uspekhi Mat. Nauk 46:4, pp 3 - 42, 1991. [English transl.: Russian Math. Surveys 46:4 (1991), pp 1 - 45.]
- [7] Bohle, Ch. *Constrained Willmore Tori in the 4-Sphere*. J. Diff. Geom., 86, pp 71-132, 2010.

- [8] Bohle, Peters and Pinkall. *Constrained Willmore Surfaces*. Calc. Var. Partial Differential Equations 32, pp 263-277, 2008.
- [9] Brendle, S. *Embedded minimal tori in S^3 and the Lawson conjecture*. arXiv:1203.6597, 2012.
- [10] Burstall, Pedit and Pinkall. *Schwarzian Derivatives and Flows of Surfaces*. Contemp. Math., 308, pp 39-61, 2002.
- [11] Burstall, Ferus, Leschke, Pedit, and Pinkall. *Conformal Geometry of Surfaces in S^4 and Quaternions*. Lecture Notes in Mathematics 1772, Springer-Verlag, Berlin, 2002.
- [12] Ferus and Pedit. *S^1 -Equivariant Minimal Tori in S^4 and S^1 -Equivariant Willmore Tori in S^3* . Mathematische Zeitschrift, Volume 204, Number 1, pp 269-282, 1990.
- [13] Heller, L. *Equivariant constrained Willmore tori in S^3* . Thesis, Eberhard Karls Universität Tübingen, 2012.
- [14] Heller, L. *Equivariant Constrained Willmore Tori in the 3-sphere*. arXiv:1211.4137, 2012.
- [15] Heller, L. *Constrained Willmore and CMC Tori in the 3-sphere*. arXiv:1212.2068, 2012.
- [16] Hsiang and Lawson. *Minimal Submanifolds of low Cohomogeneity*. J. Differential Geom. Volume 5, Number 1-2, pp 1-38, 1971.
- [17] Kuwert and Schätzle. *Minimizers of the Willmore functional under fixed conformal class*. to appear in Journal of Differential Geometry, arXiv:1009.6168v1, 2010.
- [18] Kilian, M.U. Schmidt and N. Schmitt. *Flows of constant mean curvature tori in the 3-sphere: the equivariant case*. arXiv:1011.2875v1, 2010.
- [19] Langer and Singer. *Curves in the Hyperbolic Plane and mean Curvature of Tori in 3-Space*. Bulletin of The London Mathematical Society, vol. 16, no. 5, pp. 531-534, 1984
- [20] Langer and Singer. *The Total Squared Curvature of Closed Curves*. J. Differential Geom. Volume 20, Number 1, pp 1-22, 1984.
- [21] Li and Yau. *A new conformal invariant and its applications to the Willmore conjecture and first eigenvalue of compact surfaces*. Invent. Math., 69, pp 269-291, 1982.
- [22] Marques and Neves. *Min-Max theory and the Willmore conjecture*. arXiv:1202.6036v1 2012.
- [23] Ndiaye and Schätzle. *Explicit conformally constrained Willmore minimizers in arbitrary codimension*. 2012.
- [24] Schätzle, R. *Conformally constrained Willmore immersions*. preprint 2012.
- [25] Pinkall, U. *Hopf Tori in S^3* . Invent. Math., 81(2), pp 379-386, 1985.

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